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Effective Shear Modulus for Flexural and Extensional Waves in an Unloaded Thick Plate

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Thin-plate theory suffers from failure to properly predict the correct phase speed of bending waves and extensional waves in the limit of high frequency or large thickness. Mindlin has adapted the basic ideas that Timoshenko developed for bars to the case of bending waves in plates to partly correct this situation. Kane and Mindlin have given a similar correction for extensional waves in plates. In this report a new derivation of such correction factors is presented that is based on a comparison of the approximate theory

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20. Abstract (Continued)

with the results of exact elasticity theory (Lamb waves). Explicit equations are given to compute the effective shear modulus in antisymmetric waves. It is shown analytically that the phase speed calculated with these correction factors for the shear modulus asymptotically approaches the Rayleigh wave speed for high frequency. The same principle of comparison of approximate theory with exact elasticity theory is applied to the other terms in the equations of motion besides those depending on the shear modulus. This has not been entirely successful, but it promises, among other things, to improve the identification of the second root in the quadratic dispersion relation with the first order antisymmetric and symmetric Lamb waves.

CONTENTS

LIST OF SYMBOLS

A_s	amplitude of symmetric part of potential ϕ
B_a	amplitude of antisymmetric part of potential ϕ
C_a	amplitude of antisymmetric part of potential ψ
c	phase speed
c_b	phase speed of bending waves; $c_b^2 = \dfrac{E(kd)^2}{3\rho(1-\nu^2)}$
c_d	phase speed of dilational waves; $c_d^2 = \dfrac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)}$
c_p	phase speed of extensional waves; $c_p^2 = \dfrac{E}{\rho(1-\nu^2)}$
c_R	phase speed of Rayleigh waves
c_s	phase speed of shear waves; $c_s^2 = \dfrac{E}{2\rho(1+\nu)}$
d	one-half of plate thickness
D	bending stiffness of plate; $D = \dfrac{2\ Ed^2}{3(1-\nu^2)}$
D_s	amplitude of symmetric part of potential ψ
e	$= \epsilon_x + \epsilon_y + \epsilon_z$; dilation
E	Young's modulus
f	frequency
G	shear modulus ($\equiv\mu$)
G'	effective shear modulus in antisymmetric waves
G''	effective shear modulus in symmetric waves

k	wavenumber
k_d	wavenumber of dilational wave
k_s	wavenumber of shear wave
L_x	$= \int zudz$; moment of u displacement
L_y	$= \int zvdz$; moment of v displacement
L_z	$= \int zwdz$; moment of w displacement
M_x	$= \int z\sigma_x dz$; bending moment
M_{xy}	$= \int z\sigma_{xy} dz$; twisting moment
M_y	$= \int z\sigma_y dz$; bending moment
N_x	$= \int \sigma_x dz$; stress integral
N_{xy}	$= \int \sigma_{xy} dz$; shear stress integral
N_y	$= \int \sigma_y dz$; stress integral
N_z	$= \int \sigma_z dz$; stress integral
q	$= (k^2 - k_d^2)^{1/2}$
Q_x	$= \int \sigma_{zx} dz$; shear stress integral
Q_y	$= \int \sigma_{zy} dz$; shear stress integral
r	coefficient of quadratic form in expansion of u
R_x	$= \int z\sigma_{zx} dz$; shear stress moment
R_y	$= \int z\sigma_{zy} dz$; shear stress moment
s	$(k^2 - k_s^2)^{1/2}$
$\mathbf{s}$	displacement vector
$\mathbf{s}_d$	irrotational part of $\mathbf{s}$
$\mathbf{s}_s$	solenoidal part of $\mathbf{s}$
T_x	$= \int udz$; u displacement integral
T_y	$= \int vdz$; v displacement integral

T_z	$= \int w dz$; w displacement integral
u, v, w	components of displacement vector s
U, V, W	components of zero order expansion of s in terms of z
a	$(c_s/c_d)^2 = \dfrac{1-2\nu}{2(1-\nu)}$
γ	c/c_s
γ_b	$c_b/c_s = (kd) \left(\dfrac{2}{3(1-\nu)}\right)^{1/2}$
γ_d	$c_d/c_s = \left(\dfrac{2(1-\nu)}{1-2\nu}\right)^{1/2}$
γ_p	$c_p/c_s = \left(\dfrac{2}{1-\nu}\right)^{1/2}$
γ_R	c_R/c_s
ϵ	strain tensor
κ_1^2	correction factor for shear modulus in antisymmetric waves
κ_2^2	correction factor for shear modulus in symmetric waves
κ_3^2	correction factor for bending stiffness and rotational inertia in antisymmetric waves
κ_4^2	correction factor for stiffness modulus in x-direction in symmetric waves
κ_6^2	correction factor for stiffness modulus in z-direction in symmetric waves
λ	first Lamé constant
μ	second Lamé constant ($\equiv G$)
ν	Poisson's ratio
ρ	density of solid
σ	stress tensor
Φ	$= \dfrac{\partial \phi x}{\partial x} + \dfrac{\partial \phi y}{\partial y}$

ϕ	potential for irrotational part of displacement vector
ϕ_x, ϕ_y	components of rotation angle of plate cross section
Ψ	$= \dfrac{\partial U}{\partial x} + \dfrac{\partial V}{\partial y}$
$\boldsymbol{\Psi}$	potential for solenoidal part of displacement vector
Ω	rotational displacement vector; = curl $\mathbf{s}$ = curl $\mathbf{s}_s$
ω	$= 2\pi f$; angular frequency
ψ	y component of $\boldsymbol{\Psi}$ for 2-dimensional displacement
χ	coefficient of first-order approximation of the vertical displacement w in terms of the coordinate z

EFFECTIVE SHEAR MODULUS FOR FLEXURAL AND EXTENSIONAL WAVES IN AN UNLOADED THICK PLATE

INTRODUCTION

The propagation of straight-crested waves in flat plates is an important acoustical and mechanical problem. The simplest of such problems is that of a plate in vacuum, where the plate thickness is small compared with the wavelength of a wave propagating parallel to the faces of the plate. In this case the phase speed of two types of waves can be readily derived, namely bending or flexural waves and extensional or quasi-longitudinal waves. The theory explaining these waves is indicated by thin-plate theory or classical-plate theory. On the other hand one also finds application of exact theory of elasticity to plates without restriction to small thickness, which leads to the so-called Lamb waves.

The simplicity of thin-plate theory as compared with the greater complexity of exact theory prompts one to look for corrections in those cases where thin-plate theory gives unacceptable results. Specifically, at high frequencies (or greater thicknesses), the phase speed of bending waves grows beyond bounds and the phase speed of extensional waves stays constant. Thus neither speed approaches the value for Rayleigh surface waves, as one might expect on physical grounds.

To remedy this situation, Mindlin [1] adapted the ideas of Timoshenko for propagation along bars to the case of plates, namely the introduction of the effects of transverse shear stress and rotatory inertia. In his development, Mindlin introduces an effective shear modulus to account for the difference between the simplified shear angle profile profile and the actual one. The ensuing corrective factor is fixed in such a way that the phase speed approaches the Rayleigh wave speed asymptotically for high frequencies.

Although Mindlin succeeds by this manner in devising a viable alternative to the Lamb wave solution, without the drawbacks of thin-plate theory, there is a certain unsatisfactory aspect to the determination of the correction factor in the sense that its value is imposed from outside the theory proper. This has been mentioned by other authors, for example Cremer et al. [2]. Cowper [3] also discusses various criticisms and solutions for the similar issue in the case of bars. In applications of the Mindlin-Timoshenko plate theory, it is not always appreciated that the introduction and evaluation of a correction factor depend on the physical circumstances of the problem, and the factor is certainly not a universal constant. As an example, in a paper on wave propagation in a plate loaded with an incompressible fluid layer (surface waves only), Walter and Anderson [4] state that the pertinent correction factor is the root of the dispersion relation for Rayleigh waves for a solid in vacuum. However, the dispersion relation for a fluid-loaded plate is not the same as for a plate in vacuum.

This state of affairs prompted the question of whether it would be feasible to design a new method of determining the correction factor based on the way it is introduced into the theory. Instead of choosing the factor in such a way that it reproduces the Rayleigh wave speed at high frequency, as was done by Mindlin [1], this method should

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display the proper frequency dependence of the factor in such a way that at high frequencies the Rayleigh wave speed emerges automatically. Once a method of computation is found for a correction factor operating properly for the simple case of a plate in vacuum, it is expected that this same method can be applied to more complicated cases: plates loaded by a fluid, on one or both sides; composite plates, possibly with fluid loading.

Originally the emphasis in the analysis was placed on the case of antisymmetric waves only, in connection with Timoshenko-Mindlin theory. It appears that these waves are more often referred to in hydroacoustical applications than symmetric waves. When a sound wave in a fluid impinges on a plate, both symmetric and antisymmetric waves are induced and it is necessary to know how the acoustical energy is divided over the two wave types in order to calculate the acoustic radiation, reflection, or absorption. Therefore, while embarking on a new evaluation procedure for the correction factor in the Timoshenko-Mindlin theory, it became clear that a similar procedure should be applied to the extension of the theory for extensional waves to greater plate thichnesses (or higher frequencies). An analysis of this type, from a different viewpoint, however, was performed by Kane and Mindlin [5]. Their discussion does not stress the basic analogy between the antisymmetric and symmetric waves that is an important feature of the present study. A previous example of considering Timoshenko-Mindlin and Kane-Mindlin theories simultaneously is found in the article by Walter and Anderson [4], referred to above. Their study is of limited direct value for hydroacoustics, since the fluid loading the plate is assumed incompressible and, as a consequence, only surface waves are admitted. These authors do not give an independent evaluation of the correction factors.

In the present study, the first section gives a short tutorial introduction to the subject of elasticity theory and Lamb waves. This is followed by a discussion of thin-plate theory, with the explanation of flexural and extensional waves. The subsequent derivation of plate stress equations of motion is still mainly tutorial, but emphasizes the analogy between antisymmetric and symmetric waves and their differences, based on the parity of the field variables. This approach leads to a rationale for the two types of waves. The section on thick-plate theory describes the work of Mindlin for antisymmetric waves and a parallel treatment for the case of symmetric waves.

The comparison of the thick-plate equations of motion with the corresponding equations in exact elasticity theory leads to a method of calculating a correction factor for shear modulus both in antisymmetric and symmetric waves. The resulting phase speeds possess the property desired from a viewpoint of physical intuition, namely that they asymptotically approach the phase speed of Rayleigh surface waves at high frequencies. Detailed formulae for the field variables from elasticity theory are given in Appendix A.

It would appear logical to extend this same approach to a comparison of other approximate terms in the equations of thick-plate theory with those of elasticity theory and to establish additional correction factors. Certain difficulties were encountered in this development and because of the tentative character of this theory, it is discussed in appendices B and C to the main body of the report. These other correction factors are helpful in the identification of the second root in the quadratic dispersion relation with the first mode in the antisymmetric and symmetric Lamb waves.

In the vast literature on plate theory there are various articles and reports that show an outward similarity to the work reported here. The distinction between those papers and the present analysis may be difficult to appreciate. Therefore in Appendix D, a discussion is presented of the relation of the work reported here to other work in approximate plate theory.

THEORY OF ELASTICITY APPLIED TO WAVE PROPAGATION IN PLATES

Theory of Elasticity

The basic equations of elasticity theory are derived in various texts (see e.g. Timoshenko and Goodier [6]). The relationships between the elements of the stress tensor σ and the strain tensor ϵ are given by

$$\begin{aligned} \sigma_x &= \lambda e + 2G\epsilon_x & \sigma_{xy} &= G\epsilon_{xy} \\ \sigma_y &= \lambda e + 2G\epsilon_y & \sigma_{yz} &= G\epsilon_{yz} \\ \sigma_z &= \lambda e + 2G\epsilon_z & \sigma_{zx} &= G\epsilon_{zx}, \end{aligned} \tag{1}$$

where

λ = first Lamé constant

G = shear modulus ($\equiv \mu$ second Lamé constant)

e = dilation = $\epsilon_x + \epsilon_y + \epsilon_z$.

The elements of the strain tensor are given in terms of the displacement vector **s** of a particle with components u, v, w by

$$\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} & \epsilon_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \epsilon_y &= \frac{\partial v}{\partial y} & \epsilon_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \end{aligned} \tag{2}$$

and

$$\epsilon_z = \frac{\partial w}{\partial z} \qquad \epsilon_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}.$$

The equations of motion in terms of the stress components are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = \rho \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial \sigma_{yz}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = \rho \frac{\partial^2 v}{\partial t^2} \tag{3}$$

and

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = \rho \frac{\partial^2 w}{\partial t^2},$$

where ρ is the density of the solid.

Like any vector field, the displacement field s can be represented as the sum of an irrotational field $\mathbf{s}_d$ and a solenoidal field $\mathbf{s}_s$ (Helmholtz representation). Therefore,

$$\mathbf{s} = \mathbf{s}_d + \mathbf{s}_s, \tag{4}$$

where curl $\mathbf{s}_d = 0$ and div $\mathbf{s}_s = 0$. As a consequence, for the dilation,

$$e = \text{div } \mathbf{s} = \text{div } \mathbf{s}_d; \tag{5}$$

and for the rotational displacement,

$$\Omega = \text{curl } \mathbf{s} = \text{curl } \mathbf{s}_s. \tag{6}$$

Waves related to the variations of $\mathbf{s}_d$ are indicated by dilatational waves; those connected with $\mathbf{s}_s$ are called shear waves. Only in media of infinite extent do these waves occur in a pure form: in finite media the two types mix as a consequence of the boundary conditions. In fluids only dilatational waves can occur. The terms *longitudinal* and *transverse*, respectively, have to be used with caution; a dilatational wave is longitudinal only if it is a plane wave, and a shear wave is transverse only if it is a plane wave.

The displacements in a dilatational wave can be derived from a scalar potential ϕ, where

$$\mathbf{s}_d = \text{grad } \phi, \tag{7}$$

and the displacements in a shear wave can be derived from a vector potential $\boldsymbol{\psi}$, where

$$\mathbf{s}_s = \text{curl } \boldsymbol{\psi}. \tag{8}$$

Whenever one has a 2-dimensional wave propagation, only one component of the potential $\boldsymbol{\psi}$ is nonzero. This component is indicated by ψ.

For the case of a dilatational wave the dilatation e, the displacement vector **s**, and the potential ψ are all solutions to the differential equation

$$(\lambda+2G) \nabla^2 \left\{e,s,\phi\right\} = \rho \frac{\partial^2}{\partial t^2} \left\{e,s,\phi\right\}. \tag{9}$$

Similarly for a shear wave, the relevant wave equation for the rotational displacement Ω, the displacement s, and the vector potential $\boldsymbol{\psi}$ is

$$G\nabla^2 \left\{\Omega,s,\boldsymbol{\psi}\right\} = \rho \frac{\partial^2}{\partial t^2} \left\{\Omega,s,\psi\right\}. \tag{10}$$

Notice in the above that the symbolic operator ∇^2, when operating on a vector, is given only by the Laplacian form

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{11}$$

for Cartesian coordinates. For curvilinear coordinates this operator should be interpreted as

$$\nabla^2 = \text{grad div} - \text{curl curl}. \tag{12}$$

The wave equations show that dilatational waves propagate with speed

$$c_d = [(\lambda+2G)/\rho]^{1/2}, \tag{13}$$

where c_d is the phase speed of dilatational waves. Also, shear waves propagate with speed

$$c_s = (G/\rho)^{1/2}, \tag{14}$$

where c_s is the phase speed of shear waves.

It may be pointed out here that in general the elastic properties of solids may be represented by two independent constants for which, thus far, λ and G have been used. Another useful set is Young's modulus E and Poisson's ratio ν. The following conversion will be extensively used below. The shear modulus is given by

$$G = \frac{E}{2(1+\nu)} \tag{15}$$

and the "stiffness modulus" by

$$\lambda + 2G = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}. \tag{16}$$

Lamb Waves

Lamb waves refers to the 2-dimensional straight-crested plane waves in a homogeneous, isotropic plate in the direction parallel to the faces of the plate (see Fig. 1 for the geometry). Notice that the thickness of the plate equals $2d$. Some authors use the symbol h for the plate thickness, e.g., Mindlin [1].

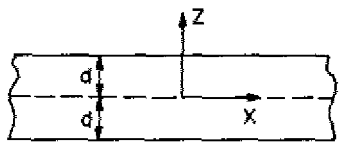

Fig. 1 — Plate geometry

The derivation of the theory of Lamb waves given below follows closely the development of Viktorov [7a].

The waves can be most advantageously expressed in terms of the potentials ϕ and ψ in the form

$$\phi = [A_s \cosh(qz) + B_a \sinh(qz)] \exp[i(\omega t \;\; kx)]$$
$$\psi = [D_s \sinh(sz) + C_a \cosh(sz)] \exp[i(\omega t \;\; kx)] \tag{17}$$

The wave propagation velocity c, the phase speed, equals ω/k, where ω is the angular frequency, k is the wavenumber, and the symbols q and s are given by

$$q^2 = k^2 - k_d^2 = k^2 \left\{1-(c/c_d)^2\right\}$$

and (18)

$$s^2 = k^2 - k_s^2 = k^2 \left\{1-(c/c_s)^2\right\},$$

where k_d and k_s are the dilatational and shear wave numbers, respectively,

$$k_d = \omega[\rho/(\lambda+2G)]^{1/2} = k(c/c_d)$$

and (19)

$$k_s = \omega(\rho/G)^{1/2} = k(c/c_s) \; .$$

The subscripts a and s in the amplitudes A_s, B_a, D_s, and C_a refer to the two possible types of Lamb waves, antisymmetric and symmetric. These terms indicate the symmetry character of the cross section of the plate (see Fig. 2). Parity of the displacement functions u and w with respect to the z coordinate is different for the two types: in antisymmetric waves u is odd and w is even, whereas in symmetric waves u is even and w is odd. The displacement components are derived from the potentials by

$$u = \frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial z}$$

and (20)

$$w = \frac{\partial\phi}{\partial z} - \frac{\partial\psi}{\partial x}.$$

This explains the various products of the amplitudes and hyperbolic functions in Eq. (17).

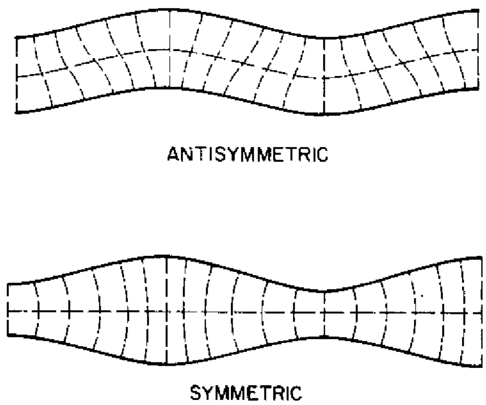

Fig. 2 — Lamb waves

The amplitudes are related through the boundary conditions, which are the values for the normal and shear stresses applied at the two faces of the plate. These stresses are found in terms of the amplitudes by means of Eqs. (1) and (2). This leads to the following set of expressions for the stresses, using the definitions of Eq. (18);

$$\begin{aligned}\sigma_z &= G\{A_s(k^2+s^2)\cosh(qz) + B_a(k^2+s^2)\sinh(qz)\\ &\quad - C_a\, 2iks\, \sinh(sz) - D_s\, 2iks\, \cosh(sz)\}\\[1em] \sigma_{zx} &= -G\{A_s\, 2ikq\, \sinh(qz) + B_a\, 2ikq\, \cosh(qz)\\ &\quad + C_a(k^2+s^2)\cosh(sz) + D_s(k^2+s^2)\sinh(sz)\}.\end{aligned} \tag{21}$$

The applied stresses for a plate in vacuum are zero. In that case the antisymmetric and symmetric waves independently satisfy the zero boundary conditions, and thus

$$B_a(k^2+s^2)\sinh(qd) - C_a\, 2iks\, \sinh(sd) = 0$$

$$B_a\, 2ikq\, \cosh(qd) + C_a(k^2+s^2)\cosh(sd) = 0$$

$$A_s(k^2+s^2)\cosh(qd) - D_s\, 2iks\, \cosh(sd) = 0$$

and (22)

$$A_s\, 2ikq\, \sinh(qd) + D_s(k^2+s^2)\sinh(sd) = 0.$$

By elimination of the amplitudes from these equations, one arrives at the dispersion relations

$$(k^2+s^2)^2 \sinh(qd)\cosh(sd) - 4k^2 qs \cosh(qd)\sinh(sd) = 0$$

for antisymmetric waves and

$$(k^2+s^2)^2 \cosh(qd)\sinh(sd) - 4k^2 qs \sinh(qd)\cosh(sd) = 0 \tag{24}$$

for symmetric waves. At large values of kd both dispersion relationships approach the same relation,

$$4k^2 qs - (k^2+s^2)^2 = 0, \tag{25}$$

which is the dispersion relation for Rayleigh waves defined as waves at the surface of a semi-infinite solid. This is understandable since, for a plate thickness that is large compared with the wavelength, the Lamb wave separates into two independent surface waves. The Rayleigh wave corresponds to the root of Eq. (25) for which the ratio c/c_s lies between 0 and 1. A good approximation for this root is the expression (Viktorov (7b))

$$\frac{c}{c_s} = \frac{0.87 + 1.12\nu}{1 + \nu} . \tag{26}$$

This ratio varies from 0.87 to 0.95 when Poisson's ratio ν varies from 0 to 0.5.

For a fixed value of the thickness $2d$ and the frequency f, the dispersion relations for Lamb waves have a finite number of real roots, corresponding to a finite number of modes of propagation in the direction of the plate. There are infinite purely imaginary roots that correspond to waves perpendicular to the faces of the plate and that decay or increase exponentially parallel to the faces of the plate. Figures 3 and 4 show the dimensionless wave speed c/c_s as a function of the dimensionless wavenumber $k_s d$ for the zero-order antisymmetric and symmetric Lamb waves, respectively.

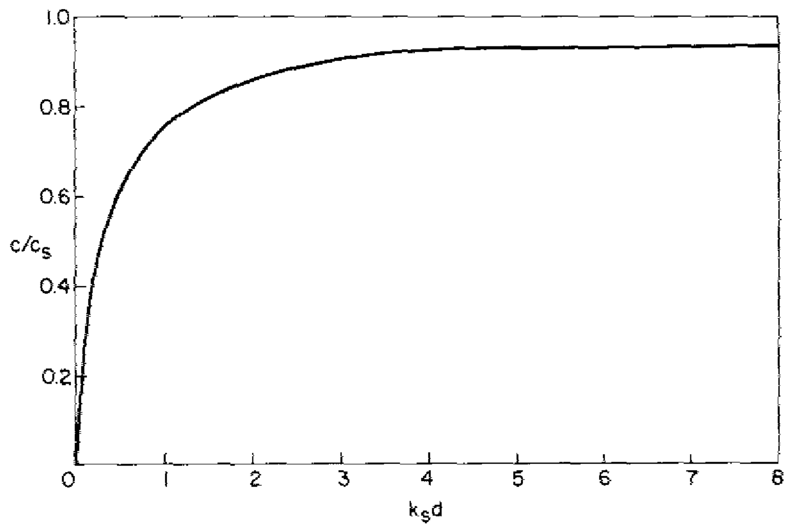

Fig. 3 — Dispersion curve for zero-order antisymmetric Lamb wave, with $\nu = 0.355$ and $\gamma_R = 0.936$

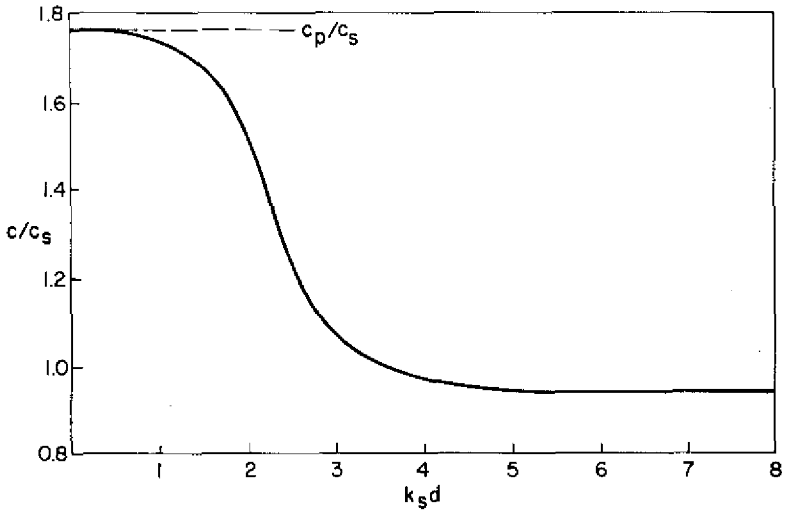

Fig. 4 — Dispersion curve for zero-order symmetric Lamb wave, with $\nu = 0.355$ and $\gamma_R = 0.936$

APPROXIMATE THEORIES FOR WAVE PROPAGATION IN PLATES

Application of the theory of elasticity to wave propagation in plates leads to rather complex expressions, expecially when composite plates and fluid loading of plates are studied. Therefore, one attempts to build a simpler theory that makes use of the fact that in plates one dimension is much smaller than the other two.

For the case where the dimensionless wavenumber kd is small, this theory explains the occurrence of two types of wave motions, flexural/bending waves and extensional waves. These two types correspond to the antisymmetric waves and symmetric waves, respectively, in elastic theory. The phase speed of bending waves is proportional to kd, and the phase speed of extensional waves is constant. Thus, in neither case does the functional dependence on kd correctly represent the behavior of the phase speed at larger kd, since it is plausible on physical grounds that the phase speed of each type approaches the phase speed of Rayleigh surface waves. To correct this flaw of the *thin-plate theory*, an extension of the theory has been developed indicated below by the term *thick-plate theory*.

Mathematically, both thin-plate and thick-plate theories can be comprehensively represented by the following series expansion of the displacement components in terms of the coordinate z to the first order (Walter and Anderson [4]):

$$\begin{aligned} u(x,y,z,t) &= U(x,y,t) + z\phi_x(x,y,t) \\ v(x,y,z,t) &= V(x,y,t) + z\phi_y(x,y,t) \\ w(x,y,z,t) &= W(x,y,t) + z\chi(x,y,t). \end{aligned} \tag{27}$$

The functions ϕ_x and ϕ_y have the character of angular rotations; the function χ represents a displacement per unit thickness.

Thin-plate theory follows by further restrictions on the functions ϕ_x, ϕ_y, and χ — restrictions that are discarded in the case of thick-plate theory. In general the same function symbols will be used for time-dependent and time-independent functions, since the dependence on time is assumed to be only in exponential form. Thus, for example, the time-dependent displacement $u(x,y,z,t)$ is alternatively expressed as $u(x,y,z)\exp(i\omega t)$.

Thin-Plate Theory: Bending (Flexural) Waves

Bending or flexural waves correspond to the antisymmetric Lamb waves. They are mathematically represented by the part of the expansion Eq. (27) for which u and v are odd functions of z and for which w is an even function of z. Thus,

$$u(x,y,z,t) = z\,\phi_x(x,y,t)$$

$$v(x,y,z,t) = z\,\phi_y(x,y,t) \tag{28}$$

and

$$w(x,y,z,t) = W(x,y,t).$$

One might surmise that bending waves are predominantly governed by shear forces, but the opposite is ture: the major elastic force is due to compressions and extensions, as in the case for extensional waves. The difference with extensional waves is that here the compression of a particle above the neutral plane is accompanied by an extension of a particle below the neutral plane, and vice versa. The thin-plate approximation emphatically excludes shear forces by making the assumption that ϵ_{xy} and ϵ_{yz} are zero, and thus the functions ϕ_x and ϕ_y are related to the function W by the equations

$$\phi_x = -\frac{\partial W}{\partial x}$$

and (29)

$$\phi_y = -\frac{\partial W}{\partial y}\,.$$

Geometrically, this means that cross sections originally perpendicular to the neutral beam will stay perpendicular to the neutral plane (see Fig. 5).

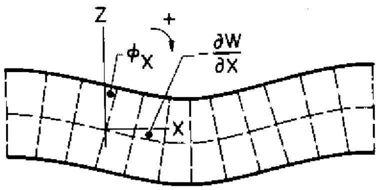

Fig. 5 — Thin-plate bending wave

In thin-plate theory it is assumed for both types of waves that zero normal stress on the faces of the plate implies that $\sigma_z(x,y,z,t) = 0$. Equation (1) shows that if σ_z is to be equal to zero, then one has $(\lambda+2G)\epsilon_z + \lambda(\epsilon_x+\epsilon_y)$ equal to zero. Eliminating ϵ_z between this expression and the expression for σ_x and σ_y in Eq. (1) results in

$$\sigma_x = \frac{E}{1-\nu^2}\,(\epsilon_x + \nu\,\epsilon_y)$$

and (30)

$$\sigma_y = \frac{E}{1-\nu^2}\,(\epsilon_y + \epsilon_x),$$

with the substitutions $\frac{4G(\lambda+G)}{\lambda + 2G} = \frac{E}{1-\nu^2}$ and $\frac{\lambda}{2(\lambda+2G)} = \nu$.

An essential feature of approximate plate theory is that the equations of motion (Eq. (3)) are integrated in the z direction, perpendicular to the faces of the plate. Depending on the parity of the displacement functions u, w with respect to the z coordinate, some expressions of the set (Eq. (3)) become identically zero upon integration. This explains the different way in which the modulus $E/(1-\nu^2)$ of Eq. (30) appears in bending waves as compared with extensional waves. The stress σ_x has odd parity with respect to z for bending waves, and as a consequence its integral over the plate thickness is equal to zero. Its role is taken over by the moment of the stress, leading to the integral $\int_{-d}^{+d} z\sigma_x dz$. With the first of the definitions (Eq. (2)), the first expression in Eq. (30), and the value for u from Eq. (28), this integral produces a "bending stiffness" D given by

$$D = \frac{2Ed^3}{3(1-\nu^2)}. \tag{31}$$

Another consequence of the introduction of moments, due to the parity of the relevant functions, is that the wave equation for bending waves is not of second order but rather fourth order, and is given by

$$D\frac{\partial^4 W}{\partial x^4} = 2\rho d\frac{\partial^2 W}{\partial t^2}, \tag{32}$$

where the function W can be replaced by any of the other field variables. (For a complete derivation see, for instance, Cremer et al. [2b].) By inserting a traveling wave expression for W into Eq. (32), one finds that bending waves are dispersive, with a dispersion relation for the wave speed c_b,

$$c_b^2 = \frac{E(kd)^2}{3\rho(1-\nu^2)}. \tag{33}$$

Thin-Plate Theory: Extensional Waves

Extensional waves correspond to symmetric Lamb waves. They are represented by the part of the expansion (Eq. (27)) for which u, v are even functions of z and for which w is an odd function of z. Thus,

$$\begin{aligned} u(x,y,z,t) &= U(x,y,t) \\ v(x,y,z,t) &= V(x,y,t) \end{aligned} \tag{34}$$

and

$$w(x,y,z,t) = z\chi(x,y,t).$$

In the case of antisymmetric waves the assumption that the shear stress σ_{zx} is zero leads to the relations of Eq. (29) among the various functions. The analogous path for symmetric waves leads from the assumption that the normal stress σ_z is zero to a relation between χ, U, and W by using the expression for σ_z in Eq. (1), namely

$$\chi = -\frac{\lambda}{\lambda+2G}\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right). \tag{35}$$

The relationships of Eq. (30) between stresses and strains in the x and y directions are also valid here. Since the parity of σ_x in the coordinate z is even, the stress integral $\int\sigma_x dz$ appears in the equations, unlike the case of bending waves where the moment of stress is needed. Thus the modulus $E/(1-\nu^2)$ in Eq. (30) determines the wave propagation, the wave equation is of second order, and the phase speed for extensional waves c_p is independent of wavelength. It is given by

$$c_p^2 = \frac{E}{\rho(1-\nu^2)}.$$

Pictorially the waves display flat cross sections moving to and fro in unison, as sketched in Fig. 6.

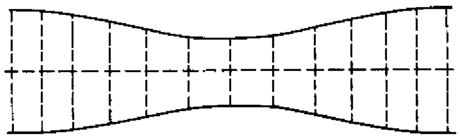

Fig. 6 — Thin-plate extensional wave

PLATE STRESS EQUATIONS OF MOTION

The development of approximate theories for wave propagation in plates is effected mathematically by integrating the equations of motion (Eq. (3)). This was mentioned before in connection with the discussion of thin-plate theory. In this section the derivation of the integrated equations is given explicitly, to serve as a basis for the understanding of the existing thick-plate theories and to give the background for evaluation of correction factors in these theories. These integrated equations are often called plate stress equations. Such an integrated set of equations does not by itself constitute a new theory or a simplification. Approximate expressions for the basic variables as given in Eqs. (27) and possible further restrictions are discussed in connection with the thin-plate theories are needed to make the mathematics more tractable.

Like any function, the displacement functions u, v, w may be written as a sum of an odd and an even part with respect to the coordinate z. Antisymmetric waves have odd u, v and even w; symmetric waves have even u, v and odd w. The parity of the derivatives of the stresses in Eq. (3), in view of Eqs. (1) and (2), is such that upon integration across the thickness direction of the plate the first two equations contain functions belonging only to the antisymmetric type, whereas the third contains functions belonging only to the symmetric type. To obtain a complete set of equations for either case requires forming integrated equations from the moment relationships, which follow from Eq. (3) by multiplying every equation by z. Thus two independent sets of equations are derived (one for each type of wave), as is shown more extensively below. If one assumes that the plate is not subject to external normal and shear stresses, the two types of waves can occur independently. Nonhomogeneous boundary conditions cause a mixture of the two types of waves. In this report only homogeneous boundary conditions are assumed.

For antisymmetric waves the moments are readily interpreted as torques of the stresses and angular momenta of the displacements. Such an identification is not obvious in the case of symmetric waves, and the moments appear only as mathematical moments in the z variable of the various functions.

Antisymmetric Waves

In this case the parity of the functions w, σ_{xz}, and σ_{yz} are even in z while u, v, σ_x, σ_y, and σ_{xy} are odd in z. Therefore, the pertinent plate stress equations are those where one takes the moment equation following from the first two parts of Eq. (3) and the force equation following from the third part of Eq. (3). As a result, one finds that

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = \rho \frac{\partial^2 L_x}{\partial t^2}$$

$$\frac{\partial M_{yx}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = \rho \frac{\partial^2 L_y}{\partial t^2} \tag{37}$$

and

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = \rho \frac{\partial^2 T_z}{\partial t^2},$$

where one introduces two bending moments M_x, M_y, one twisting moment M_{xy}, two transverse shear forces Q_x, Q_y, one integrated vertical displacement T_z, and two moments of displacement L_x, L_y. These symbols are defined by the expressions

$$M_x = \int_{-d}^{+d} z\sigma_x \mathrm{d}z \qquad M_y = \int_{-d}^{+d} z\sigma_y \mathrm{d}z$$

$$M_{yx} = M_{xy} = \int_{-d}^{+d} z\sigma_{xy} \mathrm{d}z;$$

$$Q_x = \int_{-d}^{+d} \sigma_{zx} \mathrm{d}z \qquad Q_y = \int_{-d}^{+d} \sigma_{zy} \mathrm{d}z; \tag{38}$$

$$L_x = \int_{-d}^{+d} zu \mathrm{d}z \qquad L_y = \int_{-d}^{+d} zv \mathrm{d}z;$$

and

$$T_z = \int_{-d}^{+d} w \mathrm{d}z .$$

Symmetric Waves

In this case the parity of the functions with respect to z is u, v, σ_x, σ_y, σ_z, σ_{xy} even and w, σ_{xz}, σ_{yz} odd. Therefore the pertinent plate stress equations are those obtained by the direct integration of the first two parts of Eq. (3) and from the moment equation connected with the third part of Eq. (3). One finds that

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = \rho \frac{\partial^2 T_x}{\partial t^2}$$

$$\frac{\partial N_{yz}}{\partial x} + \frac{\partial N_y}{\partial y} = \rho \frac{\partial^2 T_y}{\partial t^2} \tag{39}$$

and

$$\frac{\partial R_x}{\partial x} + \frac{\partial R_y}{\partial y} - N_z = \rho \frac{\partial^2 L_z}{\partial t^2}$$

in terms of three normal stress integrals N_x, N_y, N_z, one shear stress integral N_{xy}, and two shear stress moments R_x, R_y. Further, one needs two integrated displacements T_x, T_y and one displacement moment L_z:

$$N_x = \int_{-d}^{+d} \sigma_x \mathrm{d}z \qquad N_y = \int_{-d}^{+d} \sigma_y \mathrm{d}z$$

$$N_{xy} = \int_{-d}^{+d} \sigma_{xy} \mathrm{d}z \qquad N_z = \int_{-d}^{+d} \sigma_z \mathrm{d}z;$$

$$R_x = \int_{-d}^{+d} z\sigma_{xz} \mathrm{d}z \qquad R_y = \int_{-d}^{+d} z\sigma_{yz} \mathrm{d}z; \tag{40}$$

$$T_x = \int_{-d}^{+d} u \mathrm{d}z \qquad T_y = \int_{-d}^{+d} v \mathrm{d}z;$$

and

$$L_z = \int_{-d}^{+d} zw \mathrm{d}z.$$

THICK-PLATE THEORY

The set of integrated Eqs. (37) and (39) for wave propagation in a plate, with the definitions of Eqs. (38) and (40), do not constitute by themselves a simplification of the problem. Only if one makes assumptions concerning the displacement components u, v, w

that are plausible for a plate can one reduce these integrated equations to a form that is readily solved. Such simplifying assumptions were already given in Eq. (27). Thin-plate theory is characterized by further restrictions relating the functions ϕ_x, ϕ_y, χ to U, V, W in Eqs. (29) and (35). Models of wave propagation in plates based on the linear expansion of u, v, w in terms of z, Eq. (27), without further restrictions, are indicated by thick-plate theory. Of course it is quite conceivable that one would try to retain terms of higher order than the first in the series expansion of u, v, w to improve the accuracy of the approximation. Such higher order models have so far not come to the attention of the author (see also Appendix C).

Timoshenko-Mindlin Plate Theory

The dispersion relation for antisymmetric waves in thin plates, Eq. (33), has an unrealistic property in the sense that the phase speed increases beyond bounds as kd becomes very large, i.e., when the wavelength becomes small compared with the plate thickness. The reason for this is that the entire plate cross section in the thin-plate model is supposed to partake in a bending motion. As a consequence, the bending stiffness D in Eq. (31) is proportional to the third power of the plate thickness. The plate becomes so rigid when d increases that a disturbance is propagated instantaneously, which makes this model not acceptable on physical grounds.

Timoshenko found a means of circumventing a similar nonphysical behavior of the phase speed in the case of bars. The same basic idea of Timoshenko was applied to plates by Mindlin [1]. Physically, this idea amounts to restoring rotatory inertia to its role in the wave propagation in the plate and to dropping the requirement that cross sections originally perpendicular to the neutral plane remain so during bending. Both effects, rotatory inertia and shear strain, were ignored in the derivation of bending waves in a thin plate. Mathematically the proper equations are obtained by inserting the set of Eq. (28) into the force, moment, and displacement integrals without the restrictions of Eq. (29). This, with Eq. (30), results in

$$
\begin{aligned}
M_x &= D\left(\frac{\partial \phi_x}{\partial x} + \nu \frac{\partial \phi_y}{\partial y}\right) & M_y &= D\left(\frac{\partial \phi_y}{\partial y} + \nu \frac{\partial \phi_x}{\partial x}\right); \\
M_{xy} &= \frac{2Gd^3}{3}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right); & & \\
Q_x &= 2G'd\left(\frac{\partial W}{\partial x} + \phi_x\right) & Q_y &= 2G'\left(\frac{\partial W}{\partial y} + \phi_y\right); \\
L_x &= \frac{2d^3}{3}\phi_x & L_y &= \frac{2d^3}{3}\phi_y;
\end{aligned}
\tag{41}
$$

and

$$T_z = 2dW.$$

The symbol G' plays the role of an "effective shear modulus" and accounts for the fact that the linear expansion in z of the u, w functions is only an approximation. Without its introduction the dispersion relation, although bounded, would not approach a value plausible in the light of exact elasticity theory, namely the Rayleigh wave speed c_R. If one writes

$$G' = \kappa_1^2 G, \tag{42}$$

the effective shear modulus is represented by a dimensionless constant κ_1.

Further development of the Timoshenko-Mindlin theory consists in making the insertion of Eq. (41) into Eq. (37), differentiating the first equation of Eqs. (37) with respect to x and the second with respect to y, and then adding the results (see Ref. 8 for details). There results two equations in the quantities $\Phi = \dfrac{\partial \phi_x}{\partial x} + \dfrac{\partial \phi_y}{\partial y}$ and W;

$$D\nabla^2\Phi - 2\kappa_1^2 G d\Phi - 2\kappa_1^2 G d\nabla^2 W = \frac{2\rho d^3}{3}\frac{\partial^2\Phi}{\partial t^2}$$

and (43)

$$2\kappa_1^2 G d[\nabla^2 W + \Phi] = 2\rho d\,\frac{\partial^2 W}{\partial t^2}\ .$$

By assuming a wave solution for Φ and W of the form $\exp[i(\omega t - kx)]$, one finds the dispersion relationship

$$\begin{vmatrix} \frac{1}{3}(kd)^2 c^2 - \frac{1}{3}(kd)^2 c_p^2 - \kappa_1^2 c_s^2 & \kappa_1^2 c_s^2 \\ \kappa_1^2 c_s^2 & c^2 - \kappa_1^2 c_s^2 \end{vmatrix} = 0, \tag{44}$$

which can be written as

$$\frac{1}{3}(kd)^2(\gamma^2 - \gamma_p^2)(\gamma^2 - \kappa_1^2) = \kappa_1^2\gamma^2. \tag{45}$$

The symbol γ is used for the ratio c/c_s. The subscripts for γ follow the subscripts for the phase speed c.

This equation is quadratic in the variable γ^2. The smaller root can be identified with the zero-order Lamb wave. At low frequencies (i.e., kd close to zero) this root gives a dispersion relationship

$$\gamma^2 = \frac{1}{3}(kd)^2\gamma_p^2, \text{ or } c = c_b, \tag{46}$$

which is the dispersion relation for bending waves in a thin plate (Eq. (33)), as one would expect. This same root has a horizontal asymptote given by

$$\gamma^2 = \kappa_1^2. \tag{47}$$

This result offers Mindlin the means to fix the value of κ_1^2, since it is physically plausible that the thick-plate phase velocity would approach the phase velocity of Rayleigh waves, which are waves occurring at the surface of a semi-infinite solid. This choice of κ_1^2 does not reproduce the fact that this correction factor should be frequency dependent. As a consequence, it might be expected that the dispersion relation with this value of κ_1^2 will not be very close to the exact relation (Eq. (23)) at intermediate frequencies; but, indeed, the agreement is quite good (see Tables 1—3).

Table 1 — Dispersion Relation for Antisymmetric Waves According to Various Theories

kd	γ, I*	γ, II†	γ, III‡
0.3	0.2319	0.2384	0.2387
0.9	0.5544	0.5449	0.5478
2.3	0.7963	0.7696	0.7766
3.9	0.8607	0.8294	0.8388
5.9	0.8795	0.8517	0.8626
7.5	0.8827	0.8602	0.8703
8.7	0.8834	0.8648	0.8737
10.1	0.8836	0.8690	0.8762

*I — Exact theory.

†II — Thick-plate correction factor proposed in this report.

‡III — Thick-plate correction factor proposed by Mindlin [1].

Note: $\nu = 0.05$
$\gamma_R = 0.8837$

Table 2 — Dispersion Relation for Antisymmetric Waves According to Various Theories

kd	γ, I*	γ, II†	γ, III‡
0.3	0.2721	0.2751	0.2763
0.7	0.5338	0.5245	0.5322
1.1	0.6801	0.6614	0.6758
1.5	0.7643	0.7384	0.7575
2.5	0.8602	0.8256	0.8496
4.5	0.9122	0.8773	0.9003
6.5	0.9240	0.8948	0.9142
8.5	0.9269	0.9041	0.9197
10.1	0.9276	0.9091	0.9220

*I — Exact theory.

†II — Thick-plate correction factor proposed in this report.

‡III — Thick-plate correction factor proposed by Mindlin [1].

Note: ν = 0.3028
γ_R = 0.9278

Table 3 — Dispersion Relation for Antisymmetric Waves According to Various Theories

kd	γ, I*	γ, II†	γ, III‡
0.3	0.4827	0.3196	0.3222
0.5	0.5998	0.4759	0.4841
0.9	0.7424	0.6642	0.6854
1.3	0.8180	0.7589	0.7893
1.9	0.8771	0.8279	0.8654
2.5	0.9064	0.8619	0.9016
3.3	0.9269	0.8866	0.9256
4.3	0.9397	0.9040	0.9400
5.3	0.9465	0.9147	0.9474
6.9	0.9517	0.9254	0.9533
8.9	0.9541	0.9332	0.9570
9.9	0.9546	0.9360	0.9580

**I* — Exact theory.
†*II* — Thick-Plate correction factor proposed by this report.
‡*III* — Thick-Plate correction factor proposed by Mindlin [1].

Note: $\nu = 0.5$
$\gamma_R = 0.9553$

Mindlin [1] does not consider the larger root of Eq. (45). In the section on generalized correction factors it is shown that this root can be identified with the first-order mode in antisymmetric Lamb waves.

Symmetric Waves in Thick Plates

Although the symmetric waves in a thin plate do not have the singularity for $kd = 0$ of the bending waves, the constant phase speed c_p, given by

$$c_p = \left[\frac{E}{\rho(1-\nu^2)} \right]^{1/2},$$

does not lead to the expected asymptotic behavior, for high frequencies, namely the Rayleigh wave speed. This suggests that a correction should be developed for symmetric waves in a thin plate parallel to the correction introduced by Mindlin for antisymmetric waves in a thin plate.

Such an effort was indeed carried out by Kane and Mindlin [2] but from a different standpoint. The authors point out in their paper that, on the basis of thin-plate theory, the characteristic vibrations of a *finite* plate are determined by the lateral dimensions of the plate. If the diameter of a circular plate becomes small compared with the thickness, the thin-plate theory does not give the proper low-frequency modes, and one has to introduce thickness vibrations in order to obtain realistic values for the lowest modes of vibration. This is a different line of inquiry than the one followed in this study, where the mechanism of a thin-plate wave is considered in the transition from a true volume phenomenon toward the surface waves of Rayleigh. Therefore a further discussion of the Kane-Mindlin method will not be offered here. Instead, a development analogous to the one above for the Timoshenko-Mindlin theory will be presented, leading to the formulation of another effective shear modulus — in this case for symmetric waves in a thick plate.

One is guided to this formulation by the parallel case of antisymmetric waves, as developed in the section on Timoshenko-Mindlin plate theory. One enters the series expansion (Eq. (34)) for the displacement components in terms of the coordinate z into the integrals in Eqs. (40) that represent integrals of stresses, moments of stresses, and components of displacement. Of course, the thin-plate approximation (Eq. (35)) is not applied here. The stress components are expressed in terms of derivatives of displacement components by means of Eqs. (1). The result is a set of equations parallel to Eqs. (41) for antisymmetric waves:

$$\begin{aligned}
N_x &= 2d(\lambda+2G)\left[\frac{\partial U}{\partial x} + \left(\frac{\lambda}{\lambda+2G}\right)\left(\frac{\partial V}{\partial y} + \chi\right)\right] \\
N_y &= 2d(\lambda+2G)\left[\frac{\partial V}{\partial y} + \left(\frac{\lambda}{\lambda+2G}\right)\left(\frac{\partial U}{\partial x} + \chi\right)\right] \\
N_z &= 2d(\lambda+2G)\left[\chi + \left(\frac{\lambda}{\lambda+2G}\right)\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right)\right] \\
N_{xy} &= 2dG\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right) \\
R_x &= \frac{2}{3}d^3 G''\frac{\partial \chi}{\partial x}; \qquad R_y = \frac{2}{3}d^3 G''\frac{\partial \chi}{\partial y} \\
L_z &= \frac{2}{3}d^3\chi; \qquad T_x = 2dU; \qquad T_y = 2dV.
\end{aligned} \tag{49}$$

In searching for a correction to extensional waves in thin plates, one would be inclined to follow the successful example of the introduction of transverse shear and rotatory inertia by Mindlin for antisymmetric waves, although admittedly the latter factor had much less influence on the dispersion relation than the former. The major improvement proposed by Mindlin is the introduction of an effective shear modulus G' in the shear stress integrals Q_x and Q_y. The corresponding integrals in symmetric waves are integrals of moments of shear stress, R_x and R_y. It appears plausible to assume that a similar improvement would result from introducing another effective shear modulus G'' for symmetric waves in the expression of these integrals, as is shown in Eq. (49). The following analysis shows the validity of this choice. A nondimensional correction factor is introduced to represent this effective modulus by

$$G'' = \kappa_2^2 G. \tag{50}$$

Equations (49) and (50) are inserted into the equations of motion (Eqs. (39)). If one differentiates the first equation of motion with respect to x, and the second with respect to y, and adds the resulting equations, one obtains

$$(\lambda+2G)\nabla^2\Psi + \lambda\nabla^2\chi = \rho\frac{\partial^2\Psi}{\partial t^2}, \tag{51}$$

and the third equation of motion becomes

$$-\lambda\Psi + \frac{1}{3}\kappa_2^2 G d^2\nabla^2\chi - (\lambda+2G)\chi = \frac{1}{3}\rho d^2\frac{\partial^2\chi}{\partial t^2}, \tag{52}$$

where $\Psi = \dfrac{\partial U}{\partial x} + \dfrac{\partial V}{\partial y}$.

Assuming that Ψ and χ are represented by a straight-crested wave in the form $\exp[i(\omega t - kx)]$, one obtains the dispersion relation

$$\begin{vmatrix} \lambda+2G-\rho c^2 & \lambda \\ \lambda & \frac{1}{3}\kappa_2^2 G(kd)^2 - \frac{1}{3}(\rho c^2)(kd)^2 + \lambda + 2G \end{vmatrix} = 0. \tag{53}$$

If one introduces the various wave speeds corresponding to the elastic moduli in this equation and represents the wave speeds in dimensionless form by $\gamma = c/c_s$, then the dispersion relation appears in the alternative form,

$$\begin{vmatrix} \gamma_d^2 - \gamma^2 & \lambda_d^2 - 2 \\ \gamma_d^2 - 2 & \frac{1}{3}\kappa_2^2(kd)^2 - \frac{1}{3}\gamma^2(kd)^2 + \gamma_d^2 \end{vmatrix} = 0. \tag{54}$$

This dispersion relation is quadratic in the variable γ^2, as in the case of antisymmetric waves. The smaller root can be identified with the zero-order symmetric Lamb wave. At low frequencies (i.e., kd close to zero), this root gives for the dimensionless wavenumber

$$\gamma^2 \approx 4(1 - 1/\gamma_d^2) = \gamma_p^2. \tag{55}$$

This is the phase speed for extensional waves, as derived in the theory of symmetric waves in thin plates. This same root has a horizontal asymptote, given by

$$\gamma^2 = \kappa_2^2. \tag{56}$$

Thus by a proper choice or definition of κ_2 (as discussed below), one can ensure that the phase speed in thick-plate theory (for symmetric waves) also approaches the phase speed of Rayleigh waves. One sees, therefore, that the factor κ_2 plays a similar role in fitting the high-frequency end of the dispersion relation for symmetric waves, as the factor κ_1 did in the case of antisymmetric waves. Both are correction factors to the shear modulus G.

Just as in the case of antisymmetric waves, one can identify the second root of the dispersion relation (Eq. (54)) with the first-order mode of symmetric Lamb waves. This is discussed more extensively below.

EFFECTIVE SHEAR MODULUS IN ANTISYMMETRIC WAVES

The coefficient κ_1^2 defining the effective shear modulus with respect to the actual shear modulus in antisymmetric waves (Eq. (42)) was fixed by Mindlin [1] at a constant value such that the phase speed asymptotically approaches the Rayleigh wave speed.

It is possible to derive the value of κ_1^2 directly from theory by comparing the expression for the transverse shear force Q_x from the plate stress equations (Eqs. (37) and (38)) by utilizing Eq. (1),

$$Q_x = G \int_{-d}^{+d} \epsilon_{zx} dz, \tag{57}$$

with the corresponding expression in the thick-plate equations (Eq. (41)) and using Eq. (42),

$$Q_x = 2\kappa_1^2 \, Gd\left(\frac{\partial W}{\partial x} + \phi_x\right). \tag{58}$$

This shows that the correction factor κ_1^2 should be defined by

$$\kappa_1^2 = \frac{\dfrac{1}{d}\displaystyle\int_0^d \epsilon_{zx}\, dz}{\dfrac{\partial W}{\partial x} + \phi_x} . \tag{59}$$

The numerator in this definition can be obtained from exact elasticity theory. The denominator contains the quantities that are the field variables in thick-plate theories. To relate them to the exact theory, consider the defining equations (Eq. (28))

$$u(x,z,t) = z\phi_x(x,t)$$

and (60)

$$w(x,z,t) = W(x,t).$$

It follows that the derivative of the vertical average displacement is given as

$$\frac{\partial W}{\partial x} = \frac{1}{d}\int_0^d \frac{\partial w}{\partial x}\, dz. \tag{61}$$

The angle ϕ_x is the average angle of rotation of a plate cross section. It appears plausible to relate this average angle to the displacement component u from elasticity theory through the expression

$$\phi_x = \frac{1}{d}\int_0^d \frac{u}{z}\, dz. \tag{62}$$

This choice for ϕ_x leads to the desired asymptotic behavior of the phase speed.

The detailed formulae for the quantities in the Eqs. (59), (61), and (62), in terms of the parameters of the Lamb waves, are given in Appendix A. From Eqs. (A8), (A9), and (A10) one can infer the asymptotic values for the parts of Eq. (59), considering that for large argument the hyperbolic functions approach the exponential function. Since for $x\to\infty$, the hyperbolic integral, shi(x), defined as shi$(x) = \int_0^x \sinh(t)/t\,dt$, approaches the limit exp (x/x) (see Ref. 9), the value of σ_x in the denominator of Eq. (59) can be ignored in comparison with the value of $\partial W/\partial x$, in the high-frequency limit.

Then for $kd\to\infty$, one has

$$\frac{1}{d}\int_0^d \epsilon_{zx}\,dz \rightarrow \left(\frac{C_a}{d^2}\right)\frac{(k^2+s^2(s-q)}{qs}\exp(sd) \tag{63}$$

and

$$\frac{1}{d}\int_0^d \left(\frac{\partial w}{\partial x}\right)dz \rightarrow \left(\frac{C_a}{d^2}\right)\frac{(k^2s-2k^2q+s^3)}{2qs}\exp(sd). \tag{64}$$

As a consequence, the asymptotic value of the correction factor κ_1^2 is given by

$$\kappa_1^2 \rightarrow \frac{2(s-q)(k^2+s^2)}{k^2s+s^3-2k^2q} \text{ for } kd \rightarrow \infty. \tag{65}$$

On the other hand, the dispersion relation for Rayleigh waves derived in exact elasticity theory, Eq. (25), can be transformed into the form

$$\frac{2(s-q)(k^2+s^2)}{k^2s+s^3-2k^2q} = \frac{k^2-s^2}{k^2} \tag{66}$$

by algebraic manipulation. Since $(k^2-s^2)/k^2 = \gamma^2$ and the value of γ^2 for Rayleigh waves is the relative phase speed γ_R^2, one sees by comparing Eqs. (65) and (66) that in the limit of high frequency,

$$\kappa_1^2 \rightarrow \gamma_R^2, \text{ for } (kd) \rightarrow \infty. \tag{67}$$

This is exactly the proper high-frequency behavior of the correction factor. Here, however, the required high-frequency behavior is not imposed as in the Timoshenko-Mindlin theory, but follows naturally from the definition of κ_1^2 as given by Eq. (59), in terms of results from exact elasticity theory.

The results of calculating the correction factor κ_1 according to Eqs. (59), (61), and (62) are presented in Fig. 7. Three values of Poisson's ratio were chosen to show the variation of κ_1 as a function of this parameter. The asymptotic value, equal to the relative Rayleigh wave speed γ_R, is indicated by an arrow.

One can calculate the relative wave speed in the thick-plate approximation as a function of the relative wavenumber kd from Eq. (45). If one chooses the constant value $\kappa_1 = \gamma_R$ proposed by Mindlin, the result is quite close to the exact dispersion relation, Eq. (23). If one chooses for the correction factor κ_1 in the dispersion relation, Eq. (45), the frequency-dependent values from Fig. 7, the agreement with exact theory is less favorable, except in a few cases. This comparison is shown in Tables 1, 2, and 3.

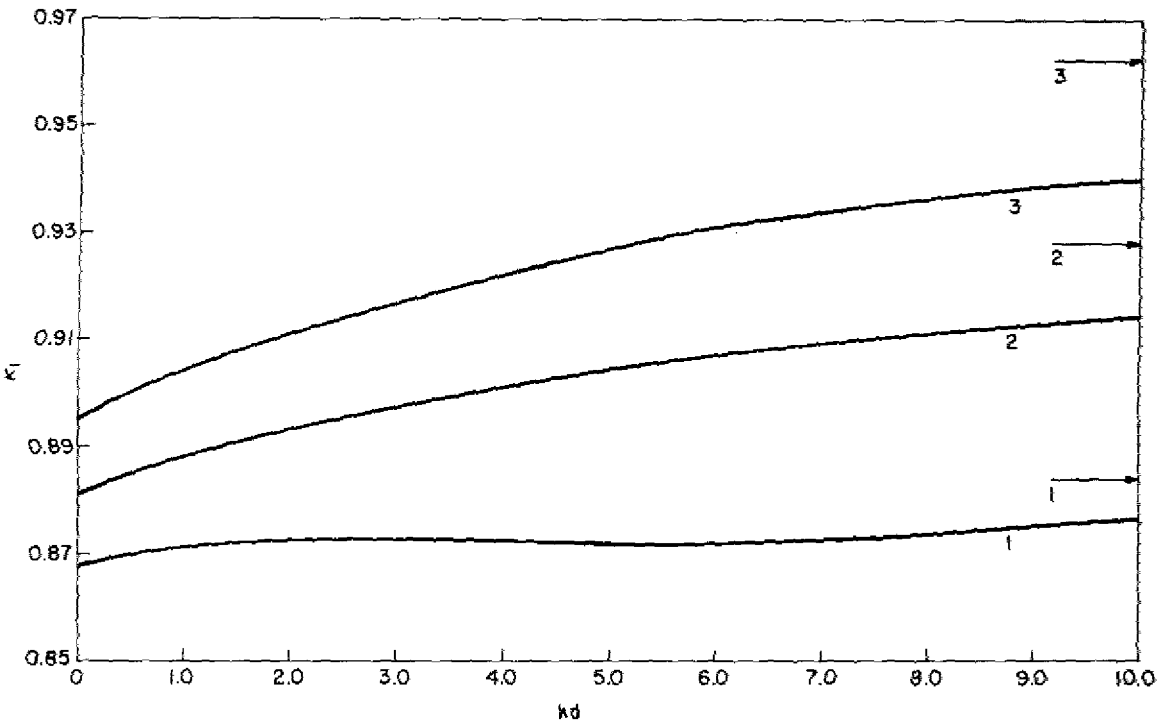

Fig. 7 — Correction factor κ_1 for effective shear modulus in antisymmetric waves; (1) $\nu = 0.05$, (2) $\nu = 0.3028$, (3) $\nu = 0.5$. The arrow → indicates the asymptote.

EFFECTIVE SHEAR MODULUS IN SYMMETRIC WAVES

It is possible to derive an expression for the effective shear modulus in symmetric waves in a way similar to the case of antisymmetric waves. Compare the expression for the stress moment integral R_x as given in Eq. (40), utilizing Eq. (1),

$$R_x = 2G\int_0^d z\epsilon_{zx}\mathrm{d}z, \tag{68}$$

with its expression as given in thick-plate theory, Eq. (49), utilizing Eq. (50),

$$R_x = \frac{2}{3}d^3\kappa_2^2 G\frac{\partial\chi}{\partial x}. \tag{69}$$

This shows that the correction factor κ_2^2 should be defined by

$$\kappa_2^2 = \frac{\dfrac{1}{d^2}\displaystyle\int_0^d z\epsilon_{zx}\,dz}{\left(\dfrac{d}{3}\right)\left(\dfrac{\partial\chi}{\partial x}\right)} \tag{70}$$

The numerator in this definition can be obtained directly from exact elasticity theory. The denominator contains the variable χ from thick-plate theory. To relate this quantity to the exact theory, consider the defining Eqs. (34),

$$u(x,z,t) = U(x,t)$$

and (71)

$$w(x,z,t) = z\chi(x,t).$$

There are several ways to define the displacement coefficient χ. The choice

$$\frac{\partial\chi}{\partial x} = \frac{3}{d^3}\int_0^d z\left(\frac{\partial w}{\partial x}\right)dz \tag{72}$$

appears plausible and, moreover, leads to the desired asymptotic behavior of the wave speed.

The detailed formulae for the quantities in Eqs. (70) and (72), in terms of the parameters of the Lamb waves, are given in Appendix A.

The numerator of Eq. (70) is given by Eq. (A19); and the denominator, as further defined by Eq. (72), is given by Eq. (A20). Equations (A19) and (A20) are obtained by integration by parts and, as a consequence, contain two parts such that the second part is smaller by one order in kd than the first part. Therefore, the second part is negligible in comparison with the first part in the limit of large kd. Since, moreover, the hyperbolic functions approach the exponential function for large kd, one sees that these first parts become equal to the formulae (Eqs. (A8) and (A9)) for antisymmetric waves, respectively. Thus, the proof given before for the asymptotic value of the correction factor in antisymmetric waves is the same for the correction factor in symmetric waves. This means that also in the latter case κ_2^2 approaches the desired value γ_R^2 for $kd \to \infty$. However, the behavior of κ_2^2 for intermediate and lower kd is unsatisfactory, in the sense that it does not stay positive. This difference in behavior as compared with the case of antisymmetric waves is due to the factors $(kd)^2 + (sd)^2$ and $(sd)^2$. The dispersion relation for symmetric waves is such that these factors become negative for low values of kd. The conclusion, therefore, has to be that the definition of κ_2^2 according to Eq. (70) combined with Eq. (72) does provide the correct high-frequency value, but it cannot be used for lower frequencies.

Efforts to improve this situation should be considered against the background of generalized correction factors. This is discussed in Appendix B.

The dispersion relation according to thick-plate theory for symmetric waves, Eq. (54), was computed using a constant value for κ equal to γ_R. The results are given in Tables 4, 5, and 6 and are compared with the dispersion relation for symmetric Lamb waves, Eq. (24). One can observe that the agreement is not as good as the corresponding case of antisymmetric waves, Tables 1 through 3.

Table 4 — Dispersion Relation for Symmetric Waves According to Various Theories

kd	I*	II†
0.1	1.4509	1.4509
0.5	1.4507	1.4508
0.9	1.4501	1.4505
1.3	1.4473	1.4499
1.7	1.3413	1.4479
2.1	1.1569	1.4347
2.5	1.0504	1.3329
2.9	0.9878	1.2340
3.3	0.9500	1.1653
3.7	0.9265	1.1138
4.5	0.9023	1.0449
5.7	0.8893	0.9873
6.9	0.8855	0.9557
8.1	0.8843	0.9365
9.7	0.8838	0.9208
13.3	—	0.9036
16.5	—	0.8967
19.7	—	0.8928

*I — Exact theory.
†II — Thick-plate theory, according to this report, with constant factor $\kappa_2 = \gamma_R$.

Note: $\nu = 0.05$
$\gamma_R = 0.8837$

Table 5 — Dispersion Relation for Symmetric Waves According to Various Theories

kd	I*	II†
0.1	1.6887	1.6933
0.5	1.6626	1.6841
0.9	1.5932	1.6601
1.3	1.4460	1.6158
1.7	1.2750	1.5465
2.1	1.1507	1.4586
2.5	1.0717	1.3698
2.9	1.0222	1.2928
3.3	0.9909	1.2305
3.7	0.9707	1.1810
4.5	0.9485	1.1106
5.7	0.9353	1.0480
6.9	0.9307	1.0124
8.1	0.9290	0.9904
9.7	0.9282	0.9722
13.3	—	0.9519
16.5	—	0.9437
19.7	—	0.9391

*I — Exact theory.

†II — Thick-plate theory, according to this report, with constant correction factor $\kappa_2 = \gamma_R$.

Note: $\nu = 0.3028$
$\gamma_R = 0.9278$

Table 6 — Dispersion Relation for Symmetric Waves According to Various Theories

kd	I*	II†
0.0	—	2.0000
0.4	1.9467	1.9607
0.8	1.8017	1.8600
1.2	1.5986	1.7329
1.6	1.4012	1.6077
2.0	1.2519	1.4978
2.4	1.1519	1.4067
2.8	1.0872	1.3330
3.2	1.0455	1.2738
3.6	1.0181	1.2264
4.4	0.9874	1.1570
5.6	0.9683	1.0930
6.8	0.9610	1.0552
8.0	0.9579	1.0314
9.6	0.9563	1.0115
13.2	—	0.9799
16.4	—	0.9747
19.6	—	0.9747

*I — Exact theory.

†II — Thick-plate theory, according to this report, with constant factor $\kappa_2 = \gamma_R$.

Note: $\nu = 0.5$

$\gamma_R = 0.9553$

CONCLUSIONS AND PLANS FOR FURTHER WORK

The main result of the study herein reported is the determination of a correction factor for effective shear modulus in antisymmetric and symmetric waves in thick plates by comparison with elasticity theory. This may be contrasted with Mindlin's [1] approach for antisymmetric waves, whereby the correction factor is fit to represent the high-frequency limit of the dispersion relation in such a way that the phase speed is equal to the Rayleigh wave speed. The correction factor proposed in this report accomplishes this without recourse to arguments outside the theory proper, as in Mindlin [1]:

it can be shown analytically that the phase speed based on this correction factor approaches the Rayleigh wave speed asymptotically. This obtains both for antisymmetric and symmetric waves.

This approach can be extended to the correction of approximate terms in the thick-plate theories other than the term representing transversal shear. Certain difficulties with singularities arise that have not been satisfactorily resolved thus far. The method promises to more accurately identify the larger root in the quadratic dispersion relation from thick-plate theory with the first higher mode of antisymmetric and symmetric Lamb waves.

In addition to further study of this generalization of the methods in this report, the next step contemplated at this time is the evaluation of correction factors for effective shear modulus in cases where the plate is subject to other than homogeneous boundary conditions: a plate loaded on both sides or one side by a fluid, a composite plate, and a composite plate under fluid loading.

ACKNOWLEDGMENTS

The suggestion by Dr. Anthony J. Rudgers to reopen the case of effective shear modulus is gratefully acknowledged. Our numerous discussions greatly contributed to the formulation of the ideas presented in this report. The report greatly benefited from a critical review by Drs. R.W. Timme and A.J. Rudgers. Supporting computations by Ms. C.M. Ruggiero are very much appreciated.

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APPENDIX A

FORMULAE FOR FIELD VARIABLES FROM ELASTICITY THEORY

The symbols used in this appendix are defined in the text of the report and also can be found alphabetically arranged in the list of symbols on pages v-viii.

The dependence on time and x-coordinate of all variables occurs through a factor $\exp[i(\omega t-kx)]$ that is not repeated in the following formulae. The amplitudes C_a and D_s have the dimension of length squared;

$$q^2 = k^2[1-(c/c_d)^2] \qquad s^2 = k^2[1-(c/c_s)^2]$$
$$c_s^2 = \frac{E}{2\rho(1+\nu)} \qquad c_d^2 = \frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)}\,. \tag{A1}$$

Antisymmetric Waves

For antisymmetric waves one uses the part of the potential ϕ that has odd parity in z, and the part of the potential ψ that has even parity in z:

$$\phi = B_a \sinh(qz) \qquad \psi = C_a \cosh(sz)\ . \tag{A2}$$

The value of c for given kd follows from the dispersion relation for homogeneous boundary conditions (Eq. (23));

$$(k^2+s^2)^2\sinh(qd)\cosh(sd)-4k^2qs\cosh(qd)\sinh(sd) = 0. \tag{A3}$$

From Eq. (22), use the relation

$$B_a 2ikq\ \cosh(qd)+C_a(k^2+s^2)\cosh(sd) = 0 \tag{A4}$$

to derive

$$\frac{u}{d} = \frac{1}{d}\left(\frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial z}\right) = \frac{C_a}{d^2}\left\{\frac{[(kd)^2+(sd)^2]\cosh(sd)\sinh(qz)-2(qd)(sd)\cosh(qd)\sinh(sz)}{2(qd)\cosh(qd)}\right\} \tag{A5}$$

$$\frac{\partial w}{\partial x} = \frac{\partial^2 \phi}{\partial x \partial z} + \frac{\partial^2 \psi}{\partial x^2} =$$

$$\frac{C_a}{d^2}\left\{\frac{[(kd)^2+(sd)^2]\cosh(sd)\cosh(qz)-2(kd)^2\cosh(qd)\cosh(sz)}{2\cosh(qd)}\right\} \tag{A6}$$

$$\epsilon_{zx} = \frac{C_a}{d^2}[(kd)^2+(sd)^2]\left[\frac{\cosh(sd)\cosh(qz)-\cosh(qd)\cosh(sz)}{\cosh(qd)}\right] \tag{A7}$$

$$\frac{1}{d}\int_0^d \epsilon_{zx} dz = \frac{C_a}{d^2}[(kd)^2+(sd)^2]\left[\frac{(sd)\cosh(sd)\sinh(qd)-(qd)\cosh(qd)\sinh(sd)}{(qd)(sd)\cosh(qd)}\right] \tag{A8}$$

$$\frac{1}{d}\int_0^d \left(\frac{\partial w}{\partial x}\right) \mathrm{d}z =$$

$$\frac{C_a}{d^2}\left\{\frac{(sd)[(kd)^2+(sd)^2]\cosh(sd)\sinh(qd)-2(kd)^2(qd)\sinh(sd)\cosh(qd)}{2(qd)(sd)\cosh(qd)}\right\} \tag{A9}$$

and

$$\frac{1}{d}\int_0^d \left(\frac{u}{z}\right) \mathrm{d}z =$$

$$\frac{C_a}{d^2}\left\{\frac{[(kd)^2+(sd)^2]\cosh(sd)\mathrm{shi}(qd)-2(qd)(sd)\cosh(qd)\mathrm{shi}(sd)}{2(qd)\cosh(qd)}\right\} . \tag{A10}$$

The hyperbolic integral shi(x) in Eq. (A10) is defined by

$$\mathrm{shi}(x) = \int_0^x \frac{\sinh t}{t}\,\mathrm{d}t \tag{A11}$$

and is computed according to the series expansion

$$\text{shi}(x) = \sum_{0}^{\infty} \frac{x^{2n+1}}{(2n+1)(2n+1)!} \,. \tag{A12}$$

Symmetric Waves

For symmetric waves one uses that part of the potential ϕ that has even parity in z, and the part of the potential ψ that has odd parity in z:

$$\phi = A_s \cosh(qz); \qquad \psi = D_s \sinh(sz)$$

The value of c for a given kd follows from the dispersion relation for homogeneous boundary conditions (Eq. (24));

$$(k^2+s^2)^2 \cosh(qd)\sinh(sd) - 4k^2 qs \sinh(qd)\cosh(sd) = 0. \tag{A14}$$

From Eq. (22), use the relation

$$A_s 2ikq \sinh(qd) + D_s(k^2+s^2)\sinh(sd) = 0 \tag{A15}$$

to derive

$$\frac{u}{d} = \frac{1}{d}\left(\frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial z}\right) = \frac{D_s}{d^2}\left\{\frac{[(kd)^2+(sd)^2]\sinh(sd)\cosh(qz) - 2(qd)(sd)\sinh(qd)\cosh(sz)}{2(qd)\sinh(qd)}\right\} \tag{A16}$$

$$\frac{\partial w}{\partial x} = \frac{\partial^2\phi}{\partial x\partial z} + \frac{\partial^2\psi}{\partial x^2} = \frac{D_s}{d^2}\left\{\frac{[(kd)^2+(sd)^2]\sinh(sd)\sinh(qz) - 2(kd)^2\sinh(qd)\sinh(sz)}{2\sinh(qd)}\right\} \tag{A17}$$

$$\epsilon_{zx} = \frac{D_s}{d^2}[(kd)^2+(sd)^2]\left[\frac{\sinh(sd)\sinh(qz) - \sinh(qd)\sinh(sz)}{\sinh(qd)}\right] \tag{A18}$$

$$\frac{1}{d^2}\int_0^d z\epsilon_{zx}\,dz = \frac{D_s}{d^2}[(kd)^2 + (sd)^2 \\ \times \left[\frac{(sd)\cosh(qd)\sinh(sd) - (qd)\sinh(qd)\cosh(sd)}{(qd)(sd)\sinh(qd)} + \frac{(qd)^2-(sd)^2}{(qd)^2(sd)^2}\sinh(sd)\right] \tag{A19}$$

$$\frac{1}{d^2}\int_0^d z\left(\frac{\partial w}{\partial x}\right)dz =$$

$$\frac{D_s}{d^2}\left[\frac{[(kd)^2+(sd)^2](sd)\cosh(qd)\sinh(sd)-2(qd)(kd)^2\sinh(qd)\cosh(sd)}{2(qd)(sd)\sinh(qd)}\right. \tag{A20}$$

$$\left.+\sinh(sd)\left\{\frac{2(qd)^2(kd)^2-(sd)^2[(kd)^2+(sd)^2]}{2(qd)^2(sd)^2}\right\}\right]$$

and

$$\int_0^d \frac{1}{z}\left(\frac{\partial u}{\partial z}\right) = D_s\left[\frac{[(kd)^2+(sd)^2]\sinh(sd)\mathrm{shi}(qd)-2(sd)^2\sinh(qd)\mathrm{shi}(sd)}{2\sinh(qd)}\right]. \tag{A21}$$

APPENDIX B

GENERALIZED CORRECTION FACTORS

The principle by which the correction factors in the foregoing sections were derived can be generalized to cover also the other terms in the equations of motion for waves in thick-plate theory, Eqs. (43) (51), and (52). Just as in defining the correction factor for shear modulus, the procedure consists in comparing the integrals defined in Eqs. (38) and (40) with the thick-plate approximations, Eqs. (41) and (49), respectively. This shows that the approximations M_x and L_x have essentially the same correction factor, while the correction factor for T_z is equal to one. Thus only one additional correction factor is needed in the case of antisymmetric waves and is defined by

$$\kappa_3^2 = \frac{\int_0^d zudz}{\left(\frac{d^3}{3}\right)\phi_x} \, . \tag{B1}$$

This correction factor is the same for the term representing bending stiffness as for the rotary inertia in Eq. (43). Applying the same procedure to the case of symmetric waves leads to two additional correction factors, one for the quantity N_x,

$$\kappa_4^2 = \frac{\left[\frac{\partial u}{\partial x} + \left(\frac{\lambda}{\lambda+2G}\right)\frac{\partial w}{\partial z}\right]}{\left[\frac{\partial U}{\partial x} + \frac{\lambda}{\lambda+2G}\chi\right]} \tag{B2}$$

and one for the quantity N_z,

$$\kappa_6^2 = \frac{\left[\frac{\partial w}{\partial z} + \left(\frac{\lambda}{\lambda+2G}\right)\frac{\partial u}{\partial x}\right]}{\left[\chi + \left(\frac{\lambda}{\lambda+2G}\right)\frac{\partial U}{\partial x}\right]} \, . \tag{B3}$$

The correction factors for T_x and L_z are equal to one, provided χ is defined as $\chi = 3/d^3 \int_{-d}^{+d} zwdz$. The factors κ_4^2 and κ_6^2 are correction factors for the stiffness modulus in the x and z directions, respectively. The subscripts of the correction factors have been chosen such that those belonging to antisymmetric waves are odd (κ_1, κ_3) and those of symmetric waves are even (κ_2, κ_4, κ_6). Inserting the correction factors κ_1^2, κ_3^2 into the equations of motion for antisymmetric waves (Eq. (43)) leads to the following dispersion relation:

$$\begin{vmatrix} \frac{1}{3}\kappa_3^2(kd)^2\gamma^2 - \frac{1}{3}\kappa_3^2(kd)^2\gamma_p^2 - \kappa_1^2 & \kappa_1^2 \\ \kappa_1^2 & \gamma^2 - \kappa_1^2 \end{vmatrix} = 0. \tag{B4}$$

The smaller root of this quadratic equation is identified with the zero-order antisymmetric Lamb wave. The larger root corresponds to the first-order Lamb wave. This wave has a vertical asymptote if one represents the dispersion relation with γ as a function of $k_s d$, where k_s is the wavenumber of the shear wave corresponding to the given frequency. By use of the relation $k_s d = \gamma k d$, Eq. (B4) is transformed into

$$\begin{vmatrix} \frac{1}{3}\kappa_3^2(k_s d)^2\gamma^2 - \kappa_1\gamma^2 - \frac{1}{3}\kappa_3^2(k_s d)^2\gamma_p^2 & \kappa_1^2\gamma^2 \\ \kappa_1^2 & \gamma^2 - \gamma_1^2 \end{vmatrix} = 0. \tag{B5}$$

It is instructive to consider the behavior of this dispersion relation, Eq. (B5), at low and at high frequencies for both roots. This is shown in Table B1. For comparison, the limits at low and at high frequencies of the dispersion relation from exact elasticity theory, Eq. (23), are also given in this table for zero-order and first-order antisymmetric Lamb waves, respectively. The comparison places constraints on the calculated values of the correction factors at low and high frequencies.

Table B1 -- Dispersion Relation for Antisymmetric Waves in the Limit of High and Low Frequencies

kd	Smaller Root	Larger Root
Small	$\gamma^2 = \kappa_1^2\gamma_b^2$ (exact theory: $\gamma^2 = \gamma_b^2$)	Asymptote for $k_s d$ at $(k_s d)^2 = 3\kappa_1^2/\kappa_3^2$ (exact theory at $(k_s d)^2 = \pi^2/4$)
Large	$\gamma^2 = \kappa_1^2$ (exact theory: $\gamma^2 = \gamma_R^2$)	$\gamma^2 = \gamma_p^2$ (exact theory: $\gamma^2 = \gamma_R^2$)

Inserting the correction factors for symmetric waves into the pertinent equations of motion, Eqs. (51) and (52), leads to the dispersion relation

$$\begin{vmatrix} \gamma^2 - \kappa_4^2\gamma_d^2 & -\kappa_4^2(\gamma_d^2 - 2) \\ \kappa_6^2(\gamma_d^2 - 2) & \frac{1}{3}\kappa_2^2(kd)^2 - \frac{1}{3}(kd)^2\gamma^2 + \kappa_6^2\gamma_d^2 \end{vmatrix} = 0. \qquad \text{(B6)}$$

The smaller root of this quadratic equation is identified with the zero-order symmetric Lamb wave. The larger root can be identified with the first-order symmetric Lamb wave. There is, again, a vertical asymptote if the dispersion curve is represented in the form of γ as a function of the dimensionless wavenumber $k_s d$. Equation (B6) is then transformed into

$$\begin{vmatrix} \gamma^2 - \kappa_4^2\gamma_d^2 & -\kappa_4^2(\gamma_d^2 - 2)\gamma^2 \\ \kappa_6^2(\gamma_d^2 - 2) & \frac{1}{3}\kappa_2^2(k_s d)^2 - \frac{1}{3}(k_s d)^2\gamma^2 + \kappa_6^2\gamma_d^2\gamma^2 \end{vmatrix} = 0. \qquad \text{(B7)}$$

By solving for the two roots of this equation, one can again derive the behavior for low and high frequencies and can compare the results with the exact theory for symmetric waves. This is shown in Table B2. If one computes the correction factor κ_3^2 according to Eq. (B1), where ϕ is determined by $\phi = (1/d)\int_0^d (u/z)dz$, an essential difficulty arises. Both numerator and denominator change sign at intermediate kd. Although the values of κ_3^2 for low and high frequencies are quite acceptable, the sign change does not occur at the same kd; as a consequence, the correction factor goes to infinity at a certain value of kd. No remedy for this anomalous behavior has been found. It is expected that the same effect may occur in the correction factors κ_4^2 and κ_6^2.

Table B2 — Dispersion Relation for Symmetric Waves in the Limit of High and Low Frequencies

kd	Smaller Root	Larger Root
Small	$\gamma^2 = \kappa_4^2\gamma_p^2$ (exact theory: $\gamma^2 = \gamma_p^2$)	Vertical asymptote at $(k_s d)^2 = 3\kappa_6^2\gamma_d^2$ (exact theory: asymptote at $(k_s d)^2 = \pi^2$)
Large	$\gamma^2 = \kappa_2^2$ (exact theory: $\gamma^2 = \gamma_R^2$)	$\gamma^2 = \kappa_4^2\gamma_d^2$ (exact theory: $\gamma^2 = \gamma_R^2$)

APPENDIX C

QUADRATIC APPROXIMATION OF DISPLACEMENT COMPONENTS

The basic mathematical model leading to thin- and thick-plate theories is the series expansion of the displacement components u, w in terms of the coordinate z, perpendicular to the phases of the plate. So far no higher terms than linear have been considered, as is shown in Eqs. (27):

$$\begin{aligned} u(x,y,z,t) &= U(x,y,t) + z\phi_x(x,y,t) \\ v(x,y,z,t) &= V(x,y,t) + z\phi_y(x,y,t) \\ w(x,y,z,t) &= W(x,y,t) + z\chi(x,y,t) \ . \end{aligned} \tag{27}$$

The problem encountered with the correction factor κ_2^2 in symmetric waves, namely that it becomes negative for low kd, prompted an effort to remedy this by inclusion of higher order terms. Thus, for symmetric waves the proposed mathematical model for the displacement coordinates would be

$$\begin{aligned} u(x,z,t) &= U(x,t)(1+rz^2/d^2), \\ w(x,z,t) &= \chi z/d, \end{aligned} \tag{C1}$$

where r is an adjustable constant. The shape of the cross section with the assumptions of Eq. (C1) is parabolic, corresponding to the visual suggestion of the symmetric wave depicted in Fig. C1.

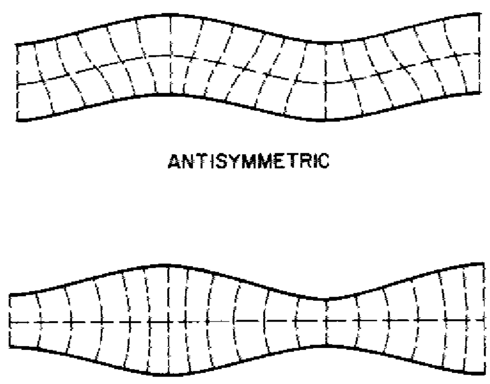

Fig. C1 — Lamb waves

Inserting Eq. (C1) into the equations of motion of Eq. (37),

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = \rho \frac{\partial^2 L_x}{\partial t^2}$$

$$\frac{\partial M_{yx}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = \rho \frac{\partial^2 L_y}{\partial t^2} \tag{37}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = \rho \frac{\partial^2 T_z}{\partial t^2},$$

with the definitions of Eq. (38)

$$M_x = \int_{-d}^{+d} z\sigma_x \mathrm{d}z \qquad M_y = \int_{-d}^{+d} z\sigma_y \mathrm{d}z$$

$$M_{yx} = M_{xy} = \int_{-d} z\sigma_{xy} \mathrm{d}z;$$

$$Q_x = \int_{-d}^{+d} \sigma_{zx} \mathrm{d}z \qquad Q_y = \int_{-d}^{+d} \sigma_{zy} \mathrm{d}z; \tag{38}$$

$$L_z = \int_{-d}^{+d} zw \mathrm{d}z$$

leads to the definition

$$\kappa_2^2 = \frac{\displaystyle\int_0^d z\epsilon_{zx} \mathrm{d}z}{\displaystyle\int_0^d z\left(\frac{\partial w}{\partial x}\right) \mathrm{d}z + \left(\frac{2}{3}\right) r \mathrm{d}U}. \tag{C2}$$

As before, there is some latitude in the choice of the new term rdU. The following definition appears plausible, and its asymptotic behavior is such that the desired value for κ_2^2 at high frequency, namely γ_R^2, as obtained before, is not influenced:

$$2rdU = d^2 \int_0^d \frac{1}{z} \left(\frac{\partial u}{\partial z} \right) \mathrm{d}z. \tag{C3}$$

This definition of κ_2^2 is an improvement in the sense that its value is positive over most of the frequency region. The same problem arises, though, as was encountered with the correction factors discussed in Appendix B, namely a sign change in the denominator that causes the value of κ_2^2 to go to infinity at a certain value of kd.

APPENDIX D

COMPARISON WITH OTHER WORK ON APPROXIMATE PLATE THEORY

In the literature one finds other work on the subject of correction factors in approximate theories for thick plates, for both antisymmetric and symetric waves. It appears that the distinction between such papers and the present analysis is not easily appreciated [D1]. The present study follows the introduction by Mindlin [D2] of a correction factor for effective shear modulus and proceeds to derive an analytic expression for this correction factor. This is different from the treatment of Mindlin, who determines the correction factor by fixing the high frequency limit of the dispersion relation by the Rayleigh wave speed. In this study, an analogous treatment is applied to symmetric waves in infinite plates and a corresponding effective shear modulus is obtained in this case. This is essentially different from the work of Kane and Mindlin [D3] and Mindlin and Medick, [D4] whose analyses apply to thickness vibrations in finite plates. Some general comments on the classification of waves in plates are in order, to clarify the distinction.

Solutions of partial differential equations, including the wave equation, are determined by the given boundary conditions. A natural way of treating waves in an infinite solid is to distinguish dilatational and shear waves, since this reflects the general property that a vector field can be represented as a sum of an irrotational and solenoidal field. In an infinite solid the two waves can exist independently. In the study of waves propagating parallel to the faces of a plate, where by definition the thickness of the plate is small compared with the dimensions parallel to the plate, the dilatational and shear waves are coupled. A logical division in this case is that of antisymmetric and symmetric waves, based on the geometric appearance of the vibrating plate. The general theory of Lamb waves is developed along this line of thought, and this is also the standpoint of the present study since its object of study is the propagation of waves in extended structures. Each of these two wave types is composed of a dilatational part and a shear part. A characteristic phenomenon in Lamb waves is the appearance of higher order modes when the frequency is increased. The physical explanation of these higher order modes is found in the occurrence of standing waves in the direction perpendicular to the faces of the plate that are coupled to the traveling waves in the direction parallel to the faces of the plate. These thickness modes can be divided into thickness-stretch and thickness-shear modes. In the thickness-stretch mode the particles move perpendicular to the faces of the plate. In the thickness-shear mode the particles move parallel to the faces of the plate. Both thickness modes may occur in an antisymmetric and a symmetric version.

One could obviously start one's analysis by emphasizing the thickness modes and treat the propagation along the plate as a secondary effect. This viewpoint is represented in a review of plate theory by Mindlin [D5] who is mostly concerned with application of the general theory to vibrations of crystal plates. It is the description of choice in the analysis of vibrations in finite plates, in contrast to the study of propagation of traveling waves in infinite plates. Therefore, it is clear that the thickness modes of vibration play a major role in Refs. D3 and D4, where finite plates are considered.

One may obtain an overview of the various wave types in plates by considering the relationships between the amplitudes for antisymmetric and symmetric waves, and the concomitant dispersion relations, similar to the discussion in Ref. D5. The following equations are identical to Eqs. (22), (23), and (24) in the main text. The equations for the amplitudes of the antisymmetric waves, under homogeneous boundary conditions, are

$$\begin{aligned} &B_a(k^2+s^2)\sinh(qd) - C_a\, 2iks\, \sinh(sd) = 0 \\ &B_a\, 2ikq\, \cosh(qd) + C_a(k^2+s^2)\cosh(sd) = 0, \end{aligned} \tag{D1}$$

with the accompanying dispersion relation for antisymmetric waves,

$$(k^2+s^2)^2\, \sinh(qd)\cosh(sd) - 4k^2 qs\, \cosh(qd)\sinh(sd) = 0 \tag{D2}$$

The equations for the amplitudes of the symmetric waves, under homogeneous boundary conditions, are

$$\begin{aligned} &A_s(k^2+s^2)^2\, \cosh(qd) - D_s\, 2iks\, \cosh(sd) = 0 \\ &A_s\, 2ikq\, \sinh(qd) + D_s(k^2+s^2)\sinh(sd) = 0, \end{aligned} \tag{D3}$$

with the accompanying dispersion relation for symmetric waves,

$$(k^2+s^2)^2\cosh(qd)\sinh(sd) - 4k^2 qs\, \sinh(qd)\cosh(sd) = 0. \tag{D4}$$

The lowest frequency at which thickness modes may occur is determined by the condition $k = 0$. This means, physically, that there is no wave propagation in the direction of the faces of the plate: the laminae of the plate move in unison, in the direction of the faces of the plate for thickness-shear modes, and perpendicular to the faces for thickness-stress modes. The propagation speeds of the pertinent waves in the direction perpendicular to the faces are the shear wave speed and the dilatational wave speed, respectively. If $k = 0$, it follows that $qd = ik_d d$ and $sd = ik_s d$. One sees, then, that, given $k = 0$, the following combinations are solutions to Eqs. (D1) through (D4). Antisymmetric thickness-shear is obtained for $B_a = 0$, $\cos(k_s d) = 0$, or $\omega = (2n + 1)(\pi/2)c_s/d$. Antisymmetric thickness-stretch modes are obtained for $C_a = 0$, $\sin(k_d d) = 0$, or $\omega = n\pi c_d/d$. Symmetric thickness shear modes occur for $A_s = 0$, $\sin(k_s d) = 0$, or $\omega = n\pi c_s/d$. Symmetric thickness-stretch modes occur for $D_s = 0$, $\cos(k_d d) = 0$, or $\omega = (2n+1)(\pi/2)c_d/d$. Here $n = 1, 2, 3...$; c_s is the shear wave speed; and c_d is the dilatational wave speed.

The structure of the Eqs. (D1) through (D4) leads to the question of what type of waves will result if the condition $k = 0$ is replaced by the condition $k^2 + s^2 = 0$. Since $s^2 = k^2 - k_s^2$, it follows that $c^2 = 2c_s^2$ for this type of wave. Given this condition, the first possibility for antisymmetric waves is that $B_a = 0$, and $\sin(k_s d/\sqrt{2}) = 0$, or $\omega = n\pi\sqrt{2}c_s/d$.. This type of wave is indicated by the term *equivoluminal* by Mindlin [D1], since the amplitude of the dilatational part of the wave is zero in this case. Actually the thickness-shear mode described before would equally well qualify for this name for the same reason. The analogous complementing case can be found by setting $C_a = 0$ and $\cosh(qd) = 0$. The latter condition cannot be satisfied, since here $q^2 = 0.5k_s^2 - k_d^2$, which is always positive. Therefore this case has no physical reality. In a similar way a symmetric "equivoluminal" wave may be derive by setting $A_s = 0$, with $\cosh(sd) = o$, which leads to $\omega = (2n+1)(\pi/2)\ \sqrt{2}c_s/d$.

In Ref. D5 Mindlin proposes a correction factor κ to improve the correspondence between approximate plate theory and exact theory. Mindlin develops his theory using the classical power-series expansion method that originated with Poisson and Cauchy. He adjusts the constant κ in such a way that the first appearance of the thickness modes occurs at the correct frequency. This leads to the value of $\pi^2/12$ for his correction factor, for both antisymmetric and symmetric waves. Since the displacement components in thickness modes vary in a harmonic way as a function of the coordinate perpendicular to the plate, it would follow that expansion in terms of a Fourier series rather than a power series promises a better correspondence between the approximate and exact theories. Employing a Fourier series expansion, however, is found to result in poor behavior of waves traveling parallel to the face of the plate at high frequencies. Again, the latter circumstance is of little importance if the object of study is vibrations of finite plates, as in Refs. D3 and D4.

In an earlier publication [D2] Mindlin had introduced a similar correction factor to improve the high-frequency behavior of the dispersion relation of the zero-order antisymmetric mode (flexural wave) in approximate plate theory. There, he so fixes the factor κ that at high frequency the wave speed approaches the Rayleigh wave speed. Mindlin comments that thus a compromise has to be found between this high-frequency adjustment of κ and the low-frequency adjustment, which amounted to setting κ equal to $\pi^2/\sqrt{12}$, as discussed before.

Here an important difference appears between the present study and the referenced papers. In the present work, the emphasis is on wave propagation along an infinite plate, represented by zero-order antisymmetric and symmetric waves, which correspond to the smaller root of the quadratic dispersion relation for flexural and extensional waves in thick-plate theory. Moreover the correction factor in the present report is not fixed at either the high-frequency or low-frequency limit, but is found by comparison of the integrals in the approximate theory with those obtained from exact elasticity theory over the whole frequency range. No "compromise" is needed, because here κ is frequency dependent, and moreover the low-frequency behavior of the zero-order mode (the smaller root of the quadratic equation) is independent of the value of κ. Thus both the high-frequency and low-frequency behaviors are properly accounted for in the present analysis.

Kane and Mindlin [D3] analyze the behavior of extensional vibrations in plates at high frequencies. One might surmise that their work is analogous to Mindlin's study of flexural motions in plates [D2] in the sense that a comparable treatment is applied to symmetric waves, but that is not the case. The study of Kane and Mindlin is limited to vibrations of finite plates, and as a consequence they fix the correction factor κ at the low-frequency end by assigning to it the familiar value $\pi^2/\sqrt{12}$. An unfortunate aspect of the basic approach of these authors is that the pertinent correction factor is introduced into the differential equations of motion of elasticity theory (Eq. (20) in Ref. D3). There is no valid reason to modify the fundamental equations of elasticity theory used all through the literature. Instead, the correction factor(s) should properly appear in the plate-stress equations, where approximations are introduced for the variation of the displacement components as a function of the coordinate perpendicular to the plate.

Mindlin and Medick [D4] expand on the ideas of Kane and Mindlin in several respects. In the first place, instead of using a straight power-series expansion, they use an expansion in Legendre polynomials for the displacement components. They expect that such a series may improve the quality of the approximation and may simplify the task of assessing the effect of truncation. In the second place, different correction factors are introduced that are similar to the correction factors of the present study. However, these authors also make the point that the correction factors can be fixed only at a finite number of points in the frequency range in order to improve the correspondence between approximate and exact theories with respect to the dispersion curves. Quite emphatically, Mindlin and Medick opt for improving the thick-plate theory at the limit of zero *kd*, in order to improve the treatment of *finite* plates, for which the thickness modes are the most important.

In connection with the above discussion, the following point should be stressed. In the study of waves in plates, one may concentrate on the propagation of traveling waves along the faces of infinite plates or on the standing waves perpendicular to the faces of the plates for finite plates. The two cases correspond to the smaller and larger roots of the quadratic equations expressing the dispersion relations in thick-plate theory. Considering the fact that the character of the motion in the two cases is quite different, there is no reason to believe that one and the same correction factor would suffice in both cases over the whole frequency range. Several correction factors are needed. Their role in connection with the low- and high-frequency behaviors is listed in Tables D1 and D2. One sees in Table D1 that the correction factor for the shear modulus in antisymmetric waves κ_1 does not influence the smaller root at low frequency. If one assumes that the factor κ_3 for the larger root has a limit of 1 at the low-frequency limit, the low-frequency limit of κ_1 for this branch will be the familiar $\pi^2/\sqrt{12}$. Remember, though, that this factor is computed here on the basis of that branch of the dispersion relation that corresponds to first-order antisymmetric Lamb waves. Table D2 shows that for symmetric waves the correction factor governing the behavior of thickness vibrations at low frequency (κ_6) is not even the same nominal factor as the one that determines the high-frequency limit for traveling waves (κ_2 in the smaller root).

Table D1 — Dispersion Relation for Antisymmetric Waves in the Limit of High and Low Frequencies

	Smaller Root	Larger Root
Small *kd*	$\gamma^2 = \kappa_3^2 \ \gamma_b^2$ (exact theory: $\gamma^2 = {\gamma_b}^2$)	Asymptote for $k_s d$ at $(k_s d)^2 = 3\kappa_1^2/\kappa_3^2$ (exact theory at $(k_s d)^2 = \pi^2/4$)
Large *kd*	$\gamma^2 = \kappa_1^2$ (exact theory: $\gamma^2 = \gamma_R^2$)	$\gamma^2 = \gamma_p^2$ (exact theory: $\gamma^2 = \gamma_R^2$)

Table D2 — Dispersion Relation for Symmetric Waves in the Limit of High and Low Frequencies

	Smaller Root	Larger Root
Small kd	$\gamma^2 = \kappa_4^2 \ \gamma_p^2$	Vertical asymptote at $(k_s d)^2 = 3\,\kappa_6^2\,\gamma_d^2$
	(exact theory: $\gamma^2 = \gamma_p^2$)	(exact theory: asymptote at $(k_s d)^2 = \pi^2$)
Large kd	$\gamma^2 = \kappa_2^2$	$\gamma^2 = \kappa_4^2\,\gamma_d^2$
	(exact theory: $\gamma^2 = \gamma_R^2$)	(exact theory: $\gamma^2 = \gamma_R^2$)

In conclusion, the present report is distinguished from Refs. D2 through D4 in the following respects.

1. It presents a thick-plate theory for plates, for which the dimensions parallel to the plate are considerably larger than the thickness, at such frequencies that the zero-order antisymmetric and symmetric Lamb waves are the only ones of importance.

2. It gives a method of calculating the correction factor for effective shear modulus in antisymmetric waves by comparing thick-plate theory with the results of exact elasticity theory, instead of fixing the correction factor at one point in the frequency range.

3. It presents a thick-plate theory for symmetric waves with a correction factor for effective shear modulus, corresponding to thick-plate theory of antisymmetric waves; it also gives a method of computing this factor by comparing thick-plate theory with the results of exact elasticity theory, instead of fixing the factor at one point in the frequency range.

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